Distribution of the residence time in a cascade of perfectly stirred vessels with backflow

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Received 9 September 1980

Accepted for publication 5 August 1981

The Klinkenberg analytical relation for calculating variances of the residence time distribution has been derived for a cascade of n perfect stirrers with backflow. This relation has been compared with some relations known from literature.

Получено аналитическое выражение Клинкенберга для расчета варианта распределения времени пребывания для каскада n идеальных смесителей с возвратным потоком. Полученное выражение сравнено с другими опубликованными соотношениями.

We assume a constant flow of incompressible liquid through a system of n perfect stirrers with backflow. The coefficient e which is the ratio of backflow to flow characterizes the magnitude of the backflow that is constant for all stirrers of the cascade (Fig. 1). Many authors were concerned with the problem of the residence time distribution in a cascade of n perfect stirrers with backflow. *Retallick* [1] solved this problem on the basis of probability of the flow of a particle through stirred vessel. *Bell* and *Babb* [2] obtained the Laplace transform of the residence time distribution from which they expressed the mean value and variance for arbitrary input signal. For the use of input signal of the Dirac δ function, the problem was solved by *Klinkenberg* [3], *Míčka* [4], and *Miyauchi* and *Vermeulen* [5].

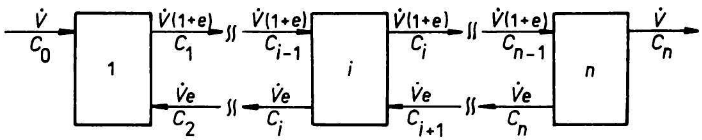

Fig. 1. Cascade of perfect stirrers with backflow.

Mathematical formulation of the problem

Let us consider a cascade of n ($n \geqslant 3$) perfectly stirred vessels with backflow arranged according to the scheme in Fig. 1. By balancing each stirrer, we obtain the following equations in dimensionless variables (in concentration-unsteady state) for $t > 0$

$$\frac{\mathrm{d}c_1}{\mathrm{d}\Theta} = c_{i-1} - (1+e)\, c_i + ec_{i+1} \qquad i = 1$$

$$\frac{\mathrm{d}c_i}{\mathrm{d}\Theta} = (1+e)\, c_{i-1} - (1+2e)\, c_i + ec_{i+1} \qquad i = 2, 3, \ldots, (n-1) \tag{1}$$

$$\frac{\mathrm{d}c_n}{\mathrm{d}\Theta} = (1+e)\, c_{i-1} - (1+e)\, c_i \qquad i = n$$

where

$$\Theta = \frac{t}{\bar{t}} \tag{1a}$$

$$\bar{t} = \frac{v}{\dot{V}} \tag{1b}$$

$$c = \frac{C}{c^*} \tag{1c}$$

$$c^* = \frac{1}{\bar{t}} \int_0^\infty C(t)\, \mathrm{d}t \tag{1d}$$

The symbol c_0 stands for the concentration input signal. In our case, the input signal was the Dirac δ function

$$c_0 = \delta(\Theta) \tag{1e}$$

The system of eqns (*1*) describing a cascade of three perfect stirrers with backflow may be transformed by successive substitution into the form

$$c_3''' + [3(1+e) + e]\, c_3'' + 3(1+e)^2 c_3' + (1+e)^2 c_3 = \\ = (1+e)^2 c_0 \tag{2}$$

Then we obtain for a cascade of four perfect stirrers with backflow

$$c_4^{(4)} + (4+6e)\, c_4''' + [6(1+e)^2 + 3e(1+e) + e^2]\, c_4'' + \\ + 4(1+e)^3 c_4' + (1+e)^3 c_4 = (1+e)^3 c_0 \tag{3}$$

By using mathematical induction for a cascade of n perfect stirrers with backflow, we can prove that it holds

$$c_n^{(n)} + A_{n-1}\, c_n^{(n-1)} + A_{n-2}\, c_n^{(n-2)} + \ldots + A_2 c_n'' + A_1 c_n' + A_0 c_n = A_0 c_0 \tag{4}$$

while

$$A_0 = (1+e)^{n-1} \tag{4a}$$

$$A_1 = n(1+e)^{n-1} \tag{4b}$$

$$A_2 = \frac{n(n-1)}{2}\, e^0(1+e)^{n-2} + \frac{(n-1)(n-2)}{2}\, e^1(1+e)^{n-3} + \\ + \ldots + \frac{3.2}{2}\, e^{n-3}(1+e)^1 + \frac{2.1}{2}\, e^{n-2}(1+e)^0 \tag{4c}$$

Writing eqn (*4*) in the Laplace variables, we obtain

$$p^n \bar{c}_n + A_{n-1} p^{n-1} \bar{c}_n + \ldots + A_2 p^2 \bar{c}_n + A_1 p \bar{c}_n + A_0 \bar{c}_n = A_0 \tag{5}$$

and on rearranging

$$\bar{c}_n = \frac{A_0}{p^n + A_{n-1} p^{n-1} + \ldots + A_2 p^2 + A_1 p + A_0} \tag{6}$$

Eqn (*6*) is the Laplace transform of the response (transfer function) to an impulse input signal for a cascade of n perfect stirrers with backflow. Eqn (*6*) enables us to calculate the variance by means of the *van der Laan* relation [6]

$$\sigma^2 = \lim_{p \to 0} \left[\frac{\mathrm{d}^2 \bar{c}_n}{\mathrm{d}p^2} - \mu^2 \right] \tag{7}$$

where

$$\mu = -\lim_{p \to 0} \frac{\mathrm{d}\bar{c}_n}{\mathrm{d}p} \tag{7a}$$

By differentiating and rearranging eqn (*6*), we obtain the expression for calculating the variance of a cascade of n perfect stirrers

$$\sigma^2 = \frac{n^2(1+e)^{n-1} - 2A_2}{(1+e)^{n-1}} \tag{8}$$

Comparison with equations proposed by other authors

Míčka [4] applied the Laplace transformation to each equation of system (*1*) and obtained the following expression by the methods of linear algebra

$$\sigma^2 = \frac{n^2(1+e)^{n-1} - Q_n}{(1+e)^{n-1}} \tag{9}$$

where

$$Q_n = \sum_{k=1}^{n} \left(R'_{n-k} \sum_{i=0}^{k-2} R_i S_{k-i-2} + R_{k-1} \sum_{i=0}^{n-k-1} S_i R'_{n-k-i-1} \right) \tag{9a}$$

and

$$R_i = \begin{vmatrix} 1+e & -e & 0 \ldots & 0 & 0 \\ -(1+2e) & 1+2e & -e \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 \ldots & -(1+e) & 1+2e \end{vmatrix} =$$

$$= (1+e)^i \qquad i = 1, 2, \ldots (n-1) \tag{9b}$$

$$R'_i = \begin{vmatrix} 1+2e & -e & 0 \ldots & 0 & 0 \\ -(1+e) & 1+2e & -e \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 \ldots & -(1+2e) & -e \\ 0 & 0 & 0 \ldots & -(1+e) & 1+e \end{vmatrix} =$$

$$= (1+e)^i \qquad i = 1, 2, \ldots (n-1) \tag{9c}$$

$$S_i = \begin{vmatrix} 1+2e & -e & 0 \ldots & 0 & 0 \\ -(1+e) & 1+2e & -e \ldots & 0 & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ 0 & 0 & 0 \ldots & -(1+e) & 1+2e \end{vmatrix} =$$

$$= (1+e)^{i+1} - e^{i+1} \qquad i = 1, 2, \ldots (n-2) \tag{9d}$$

It results from comparison of eqn (*8*) with eqn (*9*) that $Q_n \equiv A_2$.

By applying the Laplace transformation to individual equations of system (*1*) and using the statistical moments for concentrations, *Klinkenberg* [3] derived the following expression

$$\sigma^2 = n(1+2e) - 2e(1+e)\left[1 - \left(\frac{e}{1+e}\right)^n\right] \tag{10}$$

In his derivation, he evaded eqn (*7*). Our derivation is more illustrative and simpler.

We can prove by mathematical induction that eqn (*8*) modified with eqn (*4c*) and eqn (*10*) are identical for $n > 2$. For $n = 1$, it holds $A_2 = 0$ and $\sigma^2 = 1$ according to both equations. By comparing eqn (*8*) and eqn (*10*) we must prove that

$$2A_2 = n^2(1+e)^{n-1} + 2e(1+e)^n - 2e^{n+1} - n(1+e)^n - ne(1+e)^{n-1} \tag{11}$$

Eqn (*11*) is valid for $n = 2$, which may be ascertained by inserting into eqn (*4c*) which contains solely the last term and subsequently into eqn (*11*).

Let us assume that eqn (*11*) is valid for arbitrary natural number n. We must prove that it is also valid for $n+1$. If we write eqn (*11*) for $n+1$ and use that eqn (*4c*) gives

$$2A_2(n+1)=(n+1)\,n(1+e)^{n-1}+e\,2A_2(n) \qquad (12)$$

we may transform this modified equation into the form

$$(1+e)^{n+1}[(n+1)-2e]+(1+e)^n[2e^2+e-(1+n)^2]+$$
$$+(1+e)^{n-1}[n(n+1)-ne^2-n^2e]=0 \qquad (13)$$

If we raise to a power according to binomial expansion and multiply particular terms in the brackets, we obtain eight sums which, in total, represent a polynomial of the variable e. Its absolute term is to be obtained if we take the lowest index in the first, fifth, and sixth sum. Thus we obtain

$$n+1-(1+n)^2+n(n+1)=0 \qquad (14)$$

The linear term may be obtained for the lowest indices in the 1st, 2nd, 4th, 5th, 6th, and 8th sum

$$(n+1)^2e-2e+e-n(1+n^2)\,e+n(n^2-1)\,e+n^2e=0$$

Similarly, we may prove that the coefficients with e^{n+2} and e^{n+1} are equal to zero. Then we select a term e^j with arbitrary power, $j=2, 3, ..., n$. By using the properties of combination numbers, we may prove that these coefficients are also equal to zero, which makes evident the rightness of our statement.

Miyauchi and *Vermeulen* [5] put forward the equation

$$\frac{1}{2}\,\sigma^2=\frac{1}{2n}+\frac{e}{n}-\left[\left(\frac{1}{2n}+\frac{e}{n}\right)^2-\left(\frac{1}{n}\right)^2\right]\left[1-\exp\left[-\frac{\left(\frac{1}{2}+e\right)\ln\left(1+\frac{1}{e}\right)}{\frac{1}{2n}+\frac{e}{n}}\right]\right] \qquad (15)$$

where σ^2 is the variance calculated for spatial time $\bar{t}'$ which is referred to the whole volume of the system. By rewriting this equation for $\bar{t}$ defined by eqn (*1a*), we obtain

$$\sigma^2=n(1+2e)-\left[\frac{(1+2e)^2}{2}-2\right]\left[1-\exp\left(-n\ln\left(\frac{1+e}{e}\right)\right)\right] \qquad (16)$$

and further rearrangement gives

$$\sigma^2=n(1+2e)-2e(1+e)\left[1-\left(\frac{e}{1+e}\right)^n\right]+\frac{3}{2}\left[1-\left(\frac{e}{1+e}\right)^n\right] \qquad (17)$$

Eqn (*17*) gives the values of σ^2 augmented by $\frac{3}{2}\left[1-\left(\frac{e}{1+e}\right)^n\right]$ in comparison with eqn (*10*).

The greatest errors result from eqn (*17*) for small values of e and n. If the values of e increase and n does not rise over the value of 10, the absolute error expressed by eqn (*17*) decreases in comparison with eqns (*8—10*) under the value of 1% (for $n \geq 3$).

Symbols

A_i	coefficients in eqn (*5*)
C	concentration of the tracer
c^*	concentration defined by eqn (*1d*)
c	dimensionless concentration defined by eqn (*1c*)
c_0	concentration input signal
e	coefficient of backflow $\dot{V}_b \dot{V}^{-1}$
n	number of members of a cascade of perfect stirrers (CPS)
p	Laplace variable
Q	quantity defined by eqn (*9a*)
R	quantity defined by eqn (*9b*)
R'	quantity defined by eqn (*9c*)
S	quantity defined by eqn (*9d*)
$\bar{t}$	spatial time defined by eqn (*1b*)
v	volume of perfect stirrer
$\dot{V}$	liquid flow
$\dot{V}_b$	backflow
Θ	dimensionless time
$\dot{\mu}$	mean residence time
σ^2	variance of the residence time distribution
$\delta(\Theta)$	Dirac δ function

Indices

i	i-th stirrer of CPS
n	the last stirrer of a cascade of n perfect stirrers

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Translated by R. Domanský